Averaged controllability for parameter-dependent Evolution Partial Differential Equations

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1 Parameter depending control systems

Let $T > 0$. Let H and U be two Hilbert spaces. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\{A(\omega)\}_{\omega\in\Omega}$ be a family of linear operators satisfying the following conditions:

1. $A(\cdot) \in L^2(\Omega; \mathcal{L}(D(A), H))$;

2. $A(\omega) : D(A) \to H$ generates a C_0-semigroup $\{S(t, \omega)\}_{t\geq 0}$ on H for all $\omega \in \Omega$.

Let $B(\cdot) \in L^2(\Omega; \mathcal{L}(U, H))$.

Consider the following linear control system

$$\begin{cases} y_t = A(\omega)y + B(\omega)u & \text{in } (0, T], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $y_0 \in H$ and $u(\cdot) \in L^2(0, T; U)$ is the control.

In what follows, we denote by $y(\cdot, \omega; y_0)$ the solution to (1.1), which is the state of the system. Although the initial datum $y_0 \in H$ and the control $u(\cdot)$ are independent of the parameter ω, the state $y(t, \omega; y_0)$ of the system depends on ω nonlinearly.

According to the setting above, for a.e. $\omega \in \Omega$, there is a solution $y(\cdot, \omega; y_0) \in C([0, T]; H)$ and the *averaged state* (with respect to the parameter ω)

$$\int_\Omega y(\cdot, \omega; y_0)d\mu(\omega) \in C([0, T]; H).$$

Parameter-dependent evolution equations can be used to describe many uncertain physical processes (see [9, 10, 11] for example).

Several notions of controllability have been introduced but, to our best knowledge, most of them concern driving the state to a given destination by a control depending on ω (see [2, 7, 8] and the references therein). But for these properties to be useful one needs to know the specific realization of value of the uncertain parameter.

To overcome this difficulty, it is convenient to use controls which are independent of the unknown parameters. But this severely restricts our ability to handle the uncertainty of the system. Thus, we relax our goal to find a control, independent of the unknown parameters, to perform optimally in an averaged sense. This leads to the following definition:

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Definition 1.1 *System* (1.1) *is said to satisfy the property of exact averaged controllability or to be exactly controllable in average at time* T *if for any* $y_0, y_1 \in H$*, there exists a control* $u(\cdot) \in L^2(0,T;U)$ *such that the average of the solution to* (1.1) *satisfies*

$$\int_\Omega y(T,\omega;y_0)d\mu(\omega) = y_1. \tag{1.2}$$

Remark 1.1 *As in the finite dimensional context, ([12]), we can also consider the averaged control problem with parameter dependent initial data, i.e.,* $y_0 \in L^2(\Omega;H)$ *([5, Remark A.1]).*

Remark 1.2 *For parametric control systems one can also consider the problems of ensemble controllability, which concern the possibility of controlling all states with respect to different parameters simultaneously by one single control (e.g.[1, 4]). The property of exact averaged controllability we consider here is weaker than the ensemble controllability one, because it only deals with the average of the states with respect to those parameters and not with its specific realisations. The property of averaged controllability being weaker, it can be achieved in situations where ensemble controllability is impossible ([6]).*

The notion of exact averaged controllability was first introduced in [12]. In [5], the authors studied the exact averaged controllability problem for systems involving finitely-many linear parametric wave equations.

Similarly, one can also define the concepts of null averaged controllability/approximately averaged controllability.

In [6], the authors studied these controllability problems for random heat and Schrödinger equations. In particular, the following following model was analysed extensively

$$\begin{cases} y_t = \alpha(\omega)Ay + Bu & \text{in } (0,T], \\ y(0) = y_0. \end{cases} \tag{1.3}$$

Here $\alpha : \Omega \to \mathbb{R}$ is a suitable function which describes how the system depends on the parameter ω and $y_0 \in H$.

The key observation in [6] to deal with this specific parameter-dependent system was that, due to the fact that the parameter enters as a multiplier of the generator of the dynamics, the averaged dynamics can be computed in an explicit manner, allowing to identify its controllability properties. But this is impossible for more general parameter-dependent problems.

2 Some open problems on parameter-dependent averaged control

Problem (CP): *Analyze the averaged exact controllability for parametric evolution equations, as an example:*

$$\begin{cases} y_t + \alpha_1(\omega)A_1 y + \alpha_2(\omega)A_2 y = Bu & \textit{in } (0,T], \\ y(0) = y_0. \end{cases} \tag{2.1}$$

In the above, A_1 and A_2 are suitable generators of two different semigroups, and $\alpha_1, \alpha_2 : \Omega \to \mathbb{R}$ are suitable functions describing the way each of these operators enters in the system in a parametric manner.

When A_1 and A_2 commute, the techniques in [6] can be applied to (2.1) but, despite of this, the identification of the averaged dynamics and its control properties are complex and highly depend on the specific structure of $\alpha_1(\omega)$ and $\alpha_2(\omega)$.

However, in many important applications, the commutativity condition is not satisfied. An example is as follows:

$$\begin{cases} y_t - \alpha_1(\omega)\mathrm{div}\big(\sigma(x)\nabla y\big) + \alpha_2(\omega)v(x)\cdot\nabla y = \chi_{G_0}u & \text{in } G\times(0,T], \\ y = 0 & \text{on } \partial G\times(0,T], \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{2.2}$$

Here $G\subset\mathbb{R}^d$ ($d\in\mathbb{N}$) is a bounded domain with the C^2 boundary ∂G, $G_0\subset G$ is a nonempty open subset, $\sigma\in C^2(\overline{G};\mathbb{R}^{d\times d})$ and $v\in C^1(\overline{G})$.

The averaged null controllability of this parameter-dependent convection-diffusion system is a completely open challenging problem.

The same can be said about damped Schrödinger and wave equations, for instance:

$$\begin{cases} iy_t + \alpha_1(\omega)\mathrm{div}\big(\sigma(x)\nabla y\big) + i\alpha_2(\omega)v(x)y = \chi_{G_0}u & \text{in } G\times(0,T], \\ y = 0 & \text{on } \partial G\times(0,T], \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{2.3}$$

In both cases the main difficulty is to identify the dynamics governing the averaged state in the absence of control ($u\equiv 0$).

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